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12,758

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Abstract. It is known that 128 is the largest integer which is not expressible as a sum of distinct squares. Here, a computer is used to prove that 12,758 is the largest integer which is not expressible as a sum of distinct cubes.

1. Introduction. In [2], Sprague proved that every positive integer greater than 128 can be expressed as a sum of distinct perfect squares. In [3], he proved that, for every integer $n \geqq 2$, there exists a largest positive integer r_n which is not expressible as a sum of distinct nth powers (of positive integers). In light of [2], we have $r_2 = 128$. Unfortunately, the technique used in [3] is existential and does not give any idea of the size of r_n for $n > 2$. Our purpose here is twofold: (1) We use a computer to find explicitly the value of r_3 (it turns out to be 12,758), and (2) in the process, we give a new quantitative proof of Sprague's theorem for $n = 3$.

2. The Techniques. We used the following result of Richert [1]:

(*) Let $m_1, m_2, \cdots$ be an infinite increasing sequence of positive integers such that for some positive integer k the inequality $m_{i+1} \leqq 2m_i$ holds for all $i > k$. Suppose there exists a nonnegative integer a such that the numbers $a + 1, a + 2, \cdots, a + m_{k+1}$ are all expressible as sums of distinct members of the set $\{m_1, m_2, \cdots, m_k\}$. Then every integer greater than a is expressible as a sum of distinct members of the sequence $m_1, m_2, \cdots$.

Our work was done on the IBM 360/50 computer at Kansas State University. Storage problems for a large set of numbers dictated the use of the 31 binary bits of each word as storage information for 31 consecutive positive integers. (A 1 was stored if the corresponding integer is expressible as a sum of distinct cubes, a 0 if it is not.) We programmed the problem so as to obtain as much information as possible concerning the representations as sums of distinct cubes. Computer time was approximately 42 minutes.**

We ran a program which computed, for successive values of the positive integer c, the set of integers which are expressible as a sum of distinct members of the set $\{1^3, 2^3, \cdots, c^3\}$. We let c range over the interval $1 \leqq c \leqq 20$. In addition, we also obtained, in the printout, the following information: Given a positive integer b, we were told whether or not all the integers between $31b - 30$ and $31b$ inclusive were expressible as sums of distinct members of the set $\{1^3, 2^3, \cdots, 20^3\}$. From our

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** We are grateful to a referee who independently verified our result, using FORTRAN on a 6600, involving about 6 seconds of CPU time. This is a considerable improvement over our time, even considering the additional output we obtained. Our program was written in PL1.

output, we are able to study patterns of representable and nonrepresentable integers, and the number of different ways an integer can be represented. This additional information may lead to further study.

3. The Result. The following information was obtained: Each of the integers $12{,}758 + 1, 12{,}758 + 2, \cdots, 12{,}758 + 21^3$ is expressible as a sum of distinct members of the set $\{1^3, 2^3, \cdots, 20^3\}$. Thus, if we apply (*) with $m_i = i^3$, $k = 20$, and $a = 12{,}758$, we find that every integer greater than 12,758 is expressible as a sum of distinct cubes. In addition, it was a simple task, using our printout and doing the obvious arithmetic with 21^3, 22^3 and 23^3, to determine that 12,758 is not expressible as a sum of distinct cubes. Thus $r_3 = 12{,}758$.

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1. H. E. Richert, "Über Zerlegungen in paarweise verschiedene Zahlen," *Nordisk Mat. Tidskr.*, v. 31, 1949, pp. 120–122. MR **11**, 646.

2. R. Sprague, "Über Zerlegungen in ungleiche Quadratzahlen," *Math. Z.*, v. 51, 1948, pp. 289–290. MR **10**, 283.

3. R. Sprague, "Über Zerlegungen in *n*-te Potenzen mit lauter verschiedenen Grundzahlen," *Math. Z.*, v. 51, 1948, pp. 466–468. MR **10**, 514.